Tutorial Note Feb. 6

Problem 1. Assume that $f(z)=u+iv$ is differentiable. Prove that

(i) $f'(z) = u_x + iv_x = v_y - i u_y$

(ii) $f'(z) = e^{-i\theta}(u_r + iv_r)$, where $z = re^{i\theta}$, $|z| \neq 0$.

(iii) $f'(z) = -\frac{i}{z}(u_\theta + iv_\theta)$, $|z| \neq 0$.

Pf. (i) has been done during the class, by approaching z horizontally and vertically. One may apply a similar approach to prove (ii) & (iii).

(ii) $$f'(z) = \lim_{\Delta r \to 0} \frac{f((r+\Delta r)e^{i\theta}) - f(re^{i\theta})}{(r+\Delta r)e^{i\theta} - re^{i\theta}} = e^{-i\theta} \lim_{\Delta r \to 0} \frac{f((r+\Delta r)e^{i\theta}) - f(re^{i\theta})}{\Delta r}$$

$$= e^{-i\theta}(u_r + iv_r)$$

(iii) $$f'(z) = \lim_{\Delta\theta \to 0} \frac{f(re^{i(\theta+\Delta\theta)}) - f(re^{i\theta})}{re^{i(\theta+\Delta\theta)} - re^{i\theta}} = (re^{i\theta})^{-1} \lim_{\Delta\theta \to 0} \frac{f(re^{i(\theta+\Delta\theta)}) - f(re^{i\theta})}{e^{\Delta\theta i} - 1}$$

$$= z^{-1} \lim_{\Delta\theta \to 0} \frac{f(re^{i(\theta+\Delta\theta)}) - f(re^{i\theta})}{\Delta\theta} \frac{\Delta\theta}{e^{\Delta\theta i} - 1}$$

$$= z^{-1}(u_\theta + iv_\theta)\left(\frac{d}{dt} e^{it}\Big|_{t=0}\right)^{-1}$$

$$= -\frac{i}{z}(u_\theta + iv_\theta)$$

An alternative approach:

Apply change of variable formula:

$$u_x = u_r \frac{\partial r}{\partial x} + u_\theta \frac{\partial \theta}{\partial x} = u_r \cos\theta - u_\theta \frac{\sin\theta}{r}$$

and express u_y, v_x, v_y in a similar formula

Then obtain equalities by C-R equations $u_x = v_y$, $u_y = -v_x$

Express u_θ, v_θ by u_r, v_r by C-R equations to obtain (ii) and express u_r, v_r by u_θ, v_θ by C-R equations to obtain (iii).

The formulas (ii) & (iii) may help us compute the derivatives of some functions much easier. E.g. $f(z) = 1/z$.

$$f'(z) \overset{(ii)}{=} e^{-i\theta} \frac{\partial}{\partial r}\left(r^{-1}e^{-i\theta}\right) = -r^{-2}e^{-2i\theta} = -\frac{1}{z^2}$$

$$\overset{(iii)}{=} \frac{-i}{z} \frac{\partial}{\partial \theta}\left(r^{-1}e^{-i\theta}\right) = \frac{-i}{z} r^{-1}e^{-i\theta}(-i) = -\frac{1}{z^2}. \qquad \square$$

Problem 2. (i) Given $F(x,y)$, and $z = x+iy$. Prove that

$$\frac{\partial F}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right)$$

(ii) Prove that C-R equations (on u & v) $\Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0$, where $f = u + iv$

Pf. (i) $x = \frac{1}{2}(z+\bar{z})$, $y = \frac{1}{2i}(z-\bar{z})$, then

$$\frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}(F_x + iF_y)$$

(ii) $$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(f_x + if_y) = \frac{1}{2}\left(u_x + iv_x + i(u_y + iv_y)\right)$$

$$= \frac{1}{2}\left(u_x - v_y + i(v_x + u_y)\right)$$

Thus, $\frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \begin{cases} u_x = v_y \\ v_x = -u_y \end{cases}$ i.e. the C-R equation.

$\frac{\partial f}{\partial \bar{z}} = 0$ is the complex form of the C-R equation. $\square$

Problem 3. Assume that $c \in \mathbb{C}$ and $c \notin \mathbb{Z}$, find all c s.t.

(i) $|i^c|$ are all the same

(ii) i^c has only finitely many values.

<u>Sol.</u> Take $c = a+bi$, where $a, b \in \mathbb{R}$. Then

$$\log i = \ln|i| + i\arg(i) = \left(\frac{\pi}{2} + 2k\pi\right)i \quad \text{for all } k \in \mathbb{Z}$$

$$i^c = \exp(c \log i) = \exp\left((a+bi)\left(\frac{\pi}{2}+2k\pi\right)i\right)$$

$$= \exp\left(-\left(\frac{\pi}{2}+2k\pi\right)b\right)\exp\left(\left(\frac{\pi}{2}+2k\pi\right)ai\right) \qquad k \in \mathbb{Z}$$

(i) $|i^c| = \exp\left(-\left(\frac{\pi}{2}+2k\pi\right)b\right) \equiv \text{Const}$ for all $k \in \mathbb{Z}$

This is equivalent to $b = 0$.

Thus $|i^c| \equiv \text{Const}$ iff c is real.

(ii) i^c takes finitely many values $\Leftrightarrow$ both $|i^c|$ and $\operatorname{Arg}(i^c)$

takes finitely many values.

It is clear that $|i^c|$ takes finitely many values iff $b=0$.

For $\operatorname{Arg}(i^c)$, there should be k_1 & k_2 s.t. $(k_1 \neq k_2)$

$$\exp\left(\left(\frac{\pi}{2}+2k_1\pi\right)ai\right) = \exp\left(\left(\frac{\pi}{2}+2k_2\pi\right)ai\right)$$

Thus, $\exists\, k_0 \in \mathbb{Z}$ s.t. $2k_1\pi a = 2k_2\pi a + 2k_0\pi$

i.e. $a = \dfrac{k_0}{k_1 - k_2} \in \mathbb{Q}$

And it is clear that for all rational numbers c, i^c has only finitely many values.

Thus, i^c has only finitely many values iff $c \in \mathbb{Q}$ $\square$

<u>Problem 4.</u> Assume that f is analytic in a domain D and $|f| \equiv \text{Const}$.

Prove that $f \equiv \text{const}$

Pf. If $|f| \equiv 0$, we would easily derive $f \equiv 0$.

We would now assume that $|f| = c \neq 0$.

Claim: $\bar{f}$ is also analytic in D.

Notice that $f\bar{f} = |f|^2 = c^2$, and $|f| \neq 0$. we would have

$$\bar{f} = \frac{c^2}{f}, \text{ which is well-defined.}$$

Take $f = u + iv$, we have $u_x = v_y$, $u_y = -v_x$ by C-R equation

Then $\bar{f} = \dfrac{c^2}{u+iv} = \dfrac{c^2 u}{u^2+v^2} - i\dfrac{c^2 v}{u^2+v^2} \triangleq \hat{u} + i\hat{v}$

$$\hat{u}_x = \frac{(v^2-u^2)u_x - 2uvv_x}{(u^2+v^2)^2} c^2, \quad \hat{u}_y = \frac{(v^2-u^2)u_y - 2uvv_y}{(u^2+v^2)^2} c^2$$

$$\hat{v}_x = c^2 \frac{2uvu_x + (v^2-u^2)v_x}{(u^2+v^2)^2}, \quad \hat{v}_y = c^2 \frac{(v^2-u^2)v_y + 2uvu_y}{(u^2+v^2)^2}$$

Then one would have $\hat{u}_x = \hat{v}_y$, $\hat{u}_y = -\hat{v}_x$. Thus $\bar{f}$ is analytic.

Claim If f & $\bar{f}$ are both analytic in D, then $f \equiv$ const.

It follows that $Re(f) = \frac{1}{2}(f + \bar{f})$ is also analytic. One considers the C-R equation on $Re(f)$ to yield:

$$Re(f)_x = 0, \quad Re(f)_y = 0.$$

The same argument can be applied to $Im(f) = \frac{1}{2i}(f - \bar{f})$, which yields

$$Im(f)_x = 0, \quad Im(f)_y = 0.$$

It follows from the multi-dimensional calculus that

$$Re(f) \equiv \text{Const} \quad \text{and} \quad Im(f) \equiv \text{Const}.$$

Thus, we have proved that $f \equiv$ const. $\square$